

# Classes and equivalence of linear sets in $\text{PG}(1, q^n)$

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## Abstract

The equivalence problem of  $\mathbb{F}_q$ -linear sets of rank  $n$  of  $\text{PG}(1, q^n)$  is investigated, also in terms of the associated variety, projecting configurations,  $\mathbb{F}_q$ -linear blocking sets of Rédei type and MRD-codes.

## 1 Introduction

Linear sets are natural generalizations of subgeometries. Let  $\Lambda = \text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(r-1, q^n)$ , where  $W$  is a vector space of dimension  $r$  over  $\mathbb{F}_{q^n}$ . A point set  $L$  of  $\Lambda$  is said to be an  $\mathbb{F}_q$ -linear set of  $\Lambda$  of rank  $k$  if it is defined by the non-zero vectors of a  $k$ -dimensional  $\mathbb{F}_q$ -vector subspace  $U$  of  $W$ , i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

The maximum field of linearity of an  $\mathbb{F}_q$ -linear set  $L_U$  is  $\mathbb{F}_{q^t}$  if  $t$  is the largest integer such that  $L_U$  is an  $\mathbb{F}_{q^t}$ -linear set. In the recent years, starting from the paper [18] by Lunardon, linear sets have been used to construct or characterize various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [24], see also [14].

One of the most natural questions about linear sets is their equivalence. Two linear sets  $L_U$  and  $L_V$  of  $\text{PG}(r-1, q^n)$  are said to be PTL-equivalent (or simply *equivalent*) if there is an element  $\varphi$  in  $\text{PTL}(r, q^n)$  such that  $L_U^\varphi = L_V$ . In the applications it is crucial to have methods to decide whether two linear

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sets are equivalent or not. For  $f \in \Gamma\mathrm{L}(r, q^n)$  we have  $L_{U^f} = L_U^{\varphi_f}$ , where  $\varphi_f$  denotes the collineation of  $\mathrm{PG}(W, \mathbb{F}_{q^n})$  induced by  $f$ . It follows that if  $U$  and  $V$  are  $\mathbb{F}_q$ -subspaces of  $W$  belonging to the same orbit of  $\Gamma\mathrm{L}(r, q^n)$ , then  $L_U$  and  $L_V$  are equivalent. The above condition is only sufficient but not necessary to obtain equivalent linear sets. This follows also from the fact that  $\mathbb{F}_q$ -subspaces of  $W$  with different ranks can define the same linear set, for example  $\mathbb{F}_q$ -linear sets of  $\mathrm{PG}(r-1, q^n)$  of rank  $k \geq rn - n + 1$  are all the same: they coincide with  $\mathrm{PG}(r-1, q^n)$ . As it was showed recently in [6], if  $r = 2$ , then there exist  $\mathbb{F}_q$ -subspaces of  $W$  of the same rank  $n$  but on different orbits of  $\Gamma\mathrm{L}(2, q^n)$  defining the same linear set of  $\mathrm{PG}(1, q^n)$ .

Suppose that  $L_U^{\varphi_f} = L_V$  for some collineation, but there is no  $\mathbb{F}_{q^n}$ -semilinear map between  $U$  and  $V$ . Then the  $\mathbb{F}_q$ -subspaces  $U^f$  and  $V$  define the same linear set, but there is no invertible  $\mathbb{F}_{q^n}$ -semilinear map between them. This observation motivates the following definition. An  $\mathbb{F}_q$ -linear set  $L_U$  with maximum field of linearity  $\mathbb{F}_q$  is called *simple* if for each  $\mathbb{F}_q$ -subspace  $V$  of  $W$  with  $\dim_q(U) = \dim_q(V)$ ,  $L_U = L_V$  only if  $U$  and  $V$  are in the same orbit of  $\Gamma\mathrm{L}(W, \mathbb{F}_{q^n})$ . Natural examples of simple linear sets are the subgeometries (cf. [17, Theorem 2.6] and [13, Section 25.5]). In [5] it was proved that  $\mathbb{F}_q$ -linear sets of rank  $n+1$  of  $\mathrm{PG}(2, q^n)$  admitting  $(q+1)$ -secants are simple. This allowed the authors to translate the question of equivalence to the study of the orbits of the stabilizer of a subgeometry on subspaces and hence to obtain the complete classification of  $\mathbb{F}_q$ -linear blocking sets in  $\mathrm{PG}(2, q^4)$ . Until now, the only known examples of non-simple linear sets are those of pseudoregulus type of  $\mathrm{PG}(1, q^n)$  for  $n \geq 5$  and  $n \neq 6$ , see [6].

In this paper we focus on linear sets of rank  $n$  of  $\mathrm{PG}(1, q^n)$ . Such linear sets are related to  $\mathbb{F}_q$ -linear blocking sets of Rédei type, MRD-codes of size  $q^{2n}$  with minimum rank distance  $n-1$  and projections of subgeometries. We first introduce a method which can be used to find non-simple linear sets of rank  $n$  of  $\mathrm{PG}(1, q^n)$ . Let  $L_U$  be a linear set of rank  $n$  of  $\mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$  and let  $\beta$  be a non-degenerate alternating form of  $W$ . Denote by  $\perp$  the orthogonal complement map induced by  $\mathrm{Tr}_{q^n/q} \circ \beta$  on  $W$  (considered as an  $\mathbb{F}_q$ -vector space). Then  $U$  and  $U^\perp$  defines the same linear set (cf. Result 2.1) and if  $U$  and  $U^\perp$  lie on different orbits of  $\Gamma\mathrm{L}(W, \mathbb{F}_{q^n})$ , then  $L_U$  is non-simple. Using this approach we show that there are non-simple linear sets of rank  $n$  of  $\mathrm{PG}(1, q^n)$  for  $n \geq 5$ , not of pseudoregulus type (cf. Proposition 3.9). Contrary to what we expected initially, simple linear sets are harder to find. We prove that the linear set of  $\mathrm{PG}(1, q^n)$  defined by the trace function is simple (cf. Theorem 3.7). We also show that linear sets of rank  $n$  of  $\mathrm{PG}(1, q^n)$  are simple for  $n \leq 4$  (cf. Theorem 4.5).

Moreover, in  $\text{PG}(1, q^n)$  we extend the definition of simple linear sets and introduce the  $\mathcal{Z}(\Gamma\text{L})$ -class and the  $\Gamma\text{L}$ -class for linear sets of rank  $n$ . In Section 5 we point out the meaning of these classes in terms of equivalence of the associated blocking sets, MRD-codes and projecting configurations.

## 2 Definitions and preliminary results

### 2.1 Dual linear sets with respect to a symplectic polarity of a line

For  $\alpha \in \mathbb{F}_{q^n}$  and a divisor  $h$  of  $n$  we will denote by  $\text{Tr}_{q^n/q^h}(\alpha)$  the trace of  $\alpha$  over the subfield  $\mathbb{F}_{q^h}$ , that is,  $\text{Tr}_{q^n/q^h}(\alpha) = \alpha + \alpha^{q^h} + \dots + \alpha^{q^{n-h}}$ . By  $N_{q^n/q^h}(\alpha)$  we will denote the norm of  $\alpha$  over the subfield  $\mathbb{F}_{q^h}$ , that is,  $N_{q^n/q^h}(\alpha) = \alpha^{1+q^h+\dots+q^{n-h}}$ . Since in the paper we will use only norms over  $\mathbb{F}_q$ , the function  $N_{q^n/q}$  will be denoted simply by  $N$ .

Starting from a linear set  $L_U$  and using a polarity  $\tau$  of the space it is always possible to construct another linear set, which is called *dual linear set of  $L_U$  with respect to the polarity  $\tau$*  (see [24]). In particular, let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $n$  of a line  $\text{PG}(W, \mathbb{F}_{q^n})$  and let  $\beta : W \times W \rightarrow \mathbb{F}_{q^n}$  be a non-degenerate reflexive  $\mathbb{F}_{q^n}$ -sesquilinear form on the 2-dimensional vector space  $W$  over  $\mathbb{F}_{q^n}$  determining a polarity  $\tau$ . The map  $\text{Tr}_{q^n/q} \circ \beta$  is a non-degenerate reflexive  $\mathbb{F}_q$ -sesquilinear form on  $W$ , when  $W$  is regarded as a  $2n$ -dimensional vector space over  $\mathbb{F}_q$ . Let  $\perp_\beta$  and  $\perp'_\beta$  be the orthogonal complement maps defined by  $\beta$  and  $\text{Tr}_{q^n/q} \circ \beta$  on the lattices of the  $\mathbb{F}_{q^n}$ -subspaces and  $\mathbb{F}_q$ -subspaces of  $W$ , respectively. The dual linear set of  $L_U$  with respect to the polarity  $\tau$  is the  $\mathbb{F}_q$ -linear set of rank  $n$  of  $\text{PG}(W, \mathbb{F}_{q^n})$  defined by the orthogonal complement  $U^{\perp'_\beta}$  and it will be denoted by  $L_U^\tau$ . Also, up to projectively equivalence, such a linear set does not depend on  $\tau$ .

For a point  $P = \langle \mathbf{z} \rangle_{\mathbb{F}_{q^n}} \in \text{PG}(W, \mathbb{F}_{q^n})$  the *weight* of  $P$  with respect to the linear set  $L_U$  is  $w_{L_U}(P) := \dim_q(\langle \mathbf{z} \rangle_{\mathbb{F}_{q^n}} \cap U)$ . Note that when  $P \in L_U$ , then the weight depends on the subspace  $U$  and not only on the set of points of  $L_U$ . It can happen that for two  $\mathbb{F}_q$ -subspaces  $U$  and  $V$  of  $W$  we have  $L_U = L_V$  with  $w_{L_U}(P) \neq w_{L_V}(P)$ . When we write “the weight of  $P \in L_U$ ”, then we always mean  $w_{L_U}(P)$  and hence when we speak about the weight of a point, we will never omit the subscript.

**Result 2.1.** From [24, Property 2.6] (with  $r = 2$ ,  $s = 1$  and  $t = n$ ) it can be easily seen that if  $L_U$  is an  $\mathbb{F}_q$ -linear set of rank  $n$  of a line  $\text{PG}(W, \mathbb{F}_{q^n})$  and  $L_U^\tau$  is its dual linear set with respect to a polarity  $\tau$ , then  $w_{L_U^\tau}(P^\tau) = w_{L_U}(P)$

for each point  $P \in \text{PG}(W, \mathbb{F}_{q^n})$ . If  $\tau$  is a symplectic polarity of a line  $\text{PG}(W, \mathbb{F}_{q^n})$ , then  $P^\tau = P$  and hence  $L_U = L_U^\tau = L_{U^{\perp'_\beta}}$ .

## 2.2 $\mathbb{F}_q$ -linear sets of $\text{PG}(1, q^n)$ of class $r$

In this paper we investigate the equivalence of  $\mathbb{F}_q$ -linear sets of rank  $n$  of the projective line  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$ . As we have seen in the introduction, two  $\mathbb{F}_q$ -linear sets  $L_U$  and  $L_V$  of rank  $n$  of  $\text{PG}(1, q^n)$  are equivalent if there is an element  $\varphi_f$  in  $\text{P}\Gamma\text{L}(2, q^n)$  such that  $L_U^{\varphi_f} = L_{U^f} = L_V$ , where  $f \in \Gamma\text{L}(W, \mathbb{F}_{q^n})$  is the semilinear map inducing  $\varphi_f$ . Hence the first step is to determine the  $\mathbb{F}_q$ -vector subspaces of  $W$  defining the same linear set. This motivates the definition of the  $\mathcal{Z}(\Gamma\text{L})$ -class and  $\Gamma\text{L}$ -class of a linear set  $L_U$  of  $\text{PG}(1, q^n)$  (cf. Definitions 2.3 and 2.4). The next proposition relies on the characterization of functions over  $\mathbb{F}_q$  determining few directions. It states that the  $\mathbb{F}_q$ -rank of  $L_U$  of  $\text{PG}(1, q^n)$  is uniquely defined when the maximum field of linearity of  $L_U$  is  $\mathbb{F}_q$ . This will allow us to state our definitions and results without further conditions on the rank of the corresponding  $\mathbb{F}_q$ -subspaces.

**Proposition 2.2.** *Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$  of rank  $n$ . The maximum field of linearity of  $L_U$  is  $\mathbb{F}_{q^d}$ , where*

$$d = \min\{w_{L_U}(P) : P \in L_U\}.$$

*If the maximum field of linearity of  $L_U$  is  $\mathbb{F}_q$ , then the rank of  $L_U$  as an  $\mathbb{F}_q$ -linear set is uniquely defined, i.e. for each  $\mathbb{F}_q$ -subspace  $V$  of  $W$  if  $L_U = L_V$ , then  $\dim_q(V) = n$ .*

*Proof.* First assume that  $\langle(0, 1)\rangle_{\mathbb{F}_{q^n}} \notin L_U$ , i.e.  $U = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$  for some  $q$ -polynomial  $f$  over  $\mathbb{F}_{q^n}$ .

Consider the following map,  $U \rightarrow \text{PG}(2, q^n) : (x, f(x)) \mapsto \langle(x, f(x), 1)\rangle_{\mathbb{F}_{q^n}}$ . We will call this  $q$ -set of  $\text{PG}(2, q^n)$  the graph of  $f$  and we will denote it by  $G_f$ . Let  $X_0, X_1, X_2$  denote the coordinate functions in  $\text{PG}(2, q^n)$  and consider the line  $X_2 = 0$  as the line at infinity, denoted by  $\ell_\infty$ . The points of  $\ell_\infty$  are called directions, denoted by  $(m) := \langle(1, m, 0)\rangle_{\mathbb{F}_{q^n}}$  and by  $(\infty) := \langle(0, 1, 0)\rangle_{\mathbb{F}_{q^n}}$ . The set of directions determined by  $f$  is

$$D_f := \left\{ \left( \frac{f(x) - f(y)}{x - y} \right) : x, y \in \mathbb{F}_{q^n}, x \neq y \right\} = \left\{ \left( \frac{f(z)}{z} \right) : z \in \mathbb{F}_{q^n}^* \right\}.$$

It follows that  $\langle(x, f(x))\rangle_{q^n} \mapsto \langle(x, f(x), 0)\rangle_{\mathbb{F}_{q^n}}$  is a bijection between the point set of  $L_U$  and the set of directions determined by  $f$ . The point  $P_m := \langle(1, m)\rangle_{\mathbb{F}_{q^n}}$  is mapped to the direction  $(m)$ .

For each line  $\ell$  through  $(m)$  if  $\ell$  meets the graph of  $f$ , then it meets it in  $q^t$  points, where  $t = w_{L_U}(P_m)$ . Indeed, suppose that  $\ell$  meets the graph of  $f$  in  $\langle (x_0, f(x_0), 1) \rangle_{\mathbb{F}_{q^n}}$ . To obtain the number of the other points of  $\ell \cap G_f$  we have to count

$$\left| \left\{ x \in \mathbb{F}_{q^n} \setminus \{x_0\} : \frac{f(x) - f(x_0)}{x - x_0} = m \right\} \right| = \left| \left\{ z \in \mathbb{F}_{q^n}^* : \frac{f(z)}{z} = m \right\} \right|,$$

which is  $q^t - 1$ .

Let  $d = \min\{w_{L_U}(P) : P \in L_U\}$ . If  $q = p^e$ ,  $p$  prime, then  $p^{de}$  is the largest  $p$ -power such that every line meets the graph of  $f$  in a multiple of  $p^{de}$  points. Then a result on the number of direction determined by functions over  $\mathbb{F}_q$  due to Ball, Blokhuis, Brouwer, Storme and Szőnyi [2], and Ball [1] yields that either  $d = n$  and  $f(x) = \lambda x$  for some  $\lambda \in \mathbb{F}_{q^n}$ , or  $\mathbb{F}_{q^d}$  is a subfield of  $\mathbb{F}_{q^n}$  and

$$q^{n-d} + 1 \leq |D_f| \leq \frac{q^n - 1}{q^d - 1}. \quad (1)$$

Moreover, if  $q^d > 2$ , then  $f$  is  $\mathbb{F}_{q^d}$ -linear. In our case we already know that  $f$  is  $\mathbb{F}_q$ -linear, so even in the case  $q^d = 2$  it follows that  $U$  is an  $\mathbb{F}_{q^d}$ -subspace of  $W$  and hence  $L_U$  is an  $\mathbb{F}_{q^d}$ -linear set. We show that  $\mathbb{F}_{q^d}$  is the maximum field of linearity of  $L_U$ . Suppose, contrary to our claim, that  $L_U$  is  $\mathbb{F}_{q^r}$ -linear of rank  $z$  for some  $r > d$ . Then  $L_U$  is also  $\mathbb{F}_q$ -linear of rank  $rz$ . It follows that  $rz \leq n$  since otherwise  $L_U = \text{PG}(1, q^n)$ . Then for the size of  $L_U$  we get  $|L_U| \leq (q^{rz} - 1)/(q^r - 1) \leq (q^n - 1)/(q^r - 1)$ . To get a contradiction, we show that this is less than  $q^{n-d} + 1$ , which is the lower bound obtain for  $|L_U|$  in (1). After rearranging we get

$$\frac{q^n - 1}{q^r - 1} < q^{n-d} + 1 \Leftrightarrow q^{n-d}(q^d + 1) < (q^{n-d} + 1)q^r.$$

The latter inequality always holds because of  $r \geq d + 1$ . This contradiction shows  $r = n$ .

Now suppose that  $\mathbb{F}_q$  is the maximum field of linearity of  $L_U$  and let  $V$  be an  $r$ -dimensional  $\mathbb{F}_q$ -subspace of  $W$  such that  $L_U = L_V$ . We cannot have  $r > n$  since  $L_U \neq \text{PG}(1, q^n)$ . Suppose, contrary to our claim, that  $r \leq n - 1$ . Then  $|L_U| \leq (q^{n-1} - 1)/(q - 1)$  contradicting (1) which gives  $q^{n-1} + 1 \leq |L_U|$ .

Now suppose that  $\langle (0, 1) \rangle_{\mathbb{F}_{q^n}} \in L_U$ . After a suitable projectivity  $\varphi_f$  we have  $\langle (0, 1) \rangle_{\mathbb{F}_{q^n}} \notin L_{U^f}$ . Of course the maximum field of linearity of  $L_U$  and  $L_{U^f}$  coincide and for each point  $P$  of  $L_U$  we have  $w_{L_U}(P) = w_{L_{U^f}}(P^{\varphi_f})$ . Hence the first part of the theorem follows. The second part also follows

easily since  $L_U = L_V$  with  $\dim_q(U) \neq \dim_q(V)$  would yield  $L_{U^f} = L_{V^f}$  with  $\dim_q(U^f) \neq \dim_q(V^f)$ , a contradiction.  $\square$

Now we can give the following definitions of classes of an  $\mathbb{F}_q$ -linear set of a line.

**Definition 2.3.** Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$  of rank  $n$  with maximum field of linearity  $\mathbb{F}_q$ . We say that  $L_U$  is of  $\mathcal{Z}(\Gamma\text{L})$ -class  $r$  if  $r$  is the largest integer such that there exist  $\mathbb{F}_q$ -subspaces  $U_1, U_2, \dots, U_r$  of  $W$  with  $L_{U_i} = L_U$  for  $i \in \{1, 2, \dots, r\}$  and  $U_i \neq \lambda U_j$  for each  $\lambda \in \mathbb{F}_{q^n}^*$  and for each  $i \neq j$ ,  $i, j \in \{1, 2, \dots, r\}$ .

**Definition 2.4.** Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$  of rank  $n$  with maximum field of linearity  $\mathbb{F}_q$ . We say that  $L_U$  is of  $\Gamma\text{L}$ -class  $s$  if  $s$  is the largest integer such that there exist  $\mathbb{F}_q$ -subspaces  $U_1, U_2, \dots, U_s$  of  $W$  with  $L_{U_i} = L_U$  for  $i \in \{1, 2, \dots, s\}$  and there is no  $f \in \Gamma\text{L}(2, q^n)$  such that  $U_i = U_j^f$  for each  $i \neq j$ ,  $i, j \in \{1, 2, \dots, s\}$ .

Simple linear sets (cf. Section 1) of  $\text{PG}(1, q^n)$  are exactly those of  $\Gamma\text{L}$ -class one. The next propositions are easy to show.

**Proposition 2.5.** Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$  of rank  $n$  with maximum field of linearity  $\mathbb{F}_q$  and let  $P$  be a point of  $\text{PG}(1, q^n)$ . Then for each  $f \in \Gamma\text{L}(2, q^n)$  we have  $w_{L_U}(P) = w_{L_{U^f}}(P^{f^2})$ .  $\square$

**Proposition 2.6.** Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$  of rank  $n$  with maximum field of linearity  $\mathbb{F}_q$  and let  $\varphi$  be a collineation of  $\text{PG}(W, \mathbb{F}_{q^n})$ . Then  $L_U$  and  $L_U^\varphi$  have the same  $\mathcal{Z}(\Gamma\text{L})$ -class and  $\Gamma\text{L}$ -class.  $\square$

**Remark 2.7.** Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $n$  of  $\text{PG}(1, q^n)$  with  $\Gamma\text{L}$ -class  $s$  and let  $U_1, U_2, \dots, U_s$  be  $\mathbb{F}_q$ -subspaces belonging to different orbits of  $\Gamma\text{L}(2, q^n)$  and defining  $L_U$ . The  $\text{P}\Gamma\text{L}(2, q^n)$ -orbit of  $L_U$  is the set

$$\bigcup_{i=1}^s \{L_{U_i^f} : f \in \Gamma\text{L}(2, q^n)\}.$$

### 3 Examples of simple and non-simple linear sets of $\text{PG}(1, q^n)$

Let  $\mathbb{V} = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$  and let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $n$  of  $\text{PG}(1, q^n) = \text{PG}(\mathbb{V}, \mathbb{F}_{q^n})$ . We can always assume (up to a projectivity) that  $L_U$  does not contain the point  $\langle(0, 1)\rangle_{\mathbb{F}_{q^n}}$ . Then  $U = U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ , for some

$q$ -polynomial  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$  over  $\mathbb{F}_{q^n}$ . For the sake of simplicity we will write  $L_f$  instead of  $L_{U_f}$  to denote the linear set defined by  $U_f$ .

According to Result 2.1 and using the same notations as in Section 2.1 if  $L_U$  is an  $\mathbb{F}_q$ -linear set of rank  $n$  of  $\text{PG}(1, q^n)$  and  $\tau$  is a symplectic polarity, then  $U^{\perp'_\beta}$  defines the same linear set as  $U$ . Since in general  $U^{\perp'_\beta}$  and  $U$  are not equivalent under the action of the group  $\Gamma\text{L}(2, q^n)$ , simple linear sets of a line are harder to find.

Consider the non-degenerate symmetric bilinear form of  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$  defined by the following rule

$$\langle x, y \rangle := \text{Tr}_{q^n/q}(xy). \quad (2)$$

Then the *adjoint map*  $\hat{f}$  of an  $\mathbb{F}_q$ -linear map  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$  of  $\mathbb{F}_{q^n}$  (with respect to the bilinear form  $\langle, \rangle$ ) is

$$\hat{f}(x) := \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}}. \quad (3)$$

Let  $\eta : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}_{q^n}$  be the non-degenerate alternating bilinear form of  $\mathbb{V}$  defined by  $\eta((x, y), (u, v)) = xv - yu$ . Then  $\eta$  induces a symplectic polarity on the line  $\text{PG}(\mathbb{V}, \mathbb{F}_{q^n})$  and

$$\eta'((x, y), (u, v)) = \text{Tr}_{q^n/q}(\eta((x, y), (u, v))) \quad (4)$$

is a non-degenerate alternating bilinear form on  $\mathbb{V}$ , when  $\mathbb{V}$  is regarded as a  $2n$ -dimensional vector space over  $\mathbb{F}_q$ . We will always denote in the paper by  $\perp$  and  $\perp'$  the orthogonal complement maps defined by  $\eta$  and  $\eta'$  on the lattices of the  $\mathbb{F}_{q^n}$ -subspaces and the  $\mathbb{F}_q$ -subspaces of  $\mathbb{V}$ , respectively. Direct calculation shows that

$$U_f^{\perp'} = U_{\hat{f}}. \quad (5)$$

Result 2.1 and (5) allow us to slightly reformulate [3, Lemma 2.6].

**Lemma 3.1** ([3]). *Let  $L_f = \{ \langle (x, f(x)) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^* \}$  be an  $\mathbb{F}_q$ -linear set of  $\text{PG}(1, q^n)$  of rank  $n$ , with  $f(x)$  a  $q$ -polynomial over  $\mathbb{F}_{q^n}$ , and let  $\hat{f}$  be the adjoint of  $f$  with respect to the bilinear form (2). Then for each point  $P \in \text{PG}(1, q^n)$  we have  $w_{L_f}(P) = w_{L_{\hat{f}}}(P)$ . In particular,  $L_f = L_{\hat{f}}$  and the maps defined by  $f(x)/x$  and  $\hat{f}(x)/x$  have the same image.*

**Lemma 3.2.** *Let  $\varphi$  be an  $\mathbb{F}_q$ -linear map of  $\mathbb{F}_{q^n}$  and for  $\lambda \in \mathbb{F}_{q^n}^*$  let  $\varphi_\lambda$  denote the  $\mathbb{F}_q$ -linear map:  $x \mapsto \varphi(\lambda x)/\lambda$ . Then for each point  $P \in \text{PG}(1, q^n)$  we have  $w_{L_\varphi}(P) = w_{L_{\varphi_\lambda}}(P)$ . In particular,  $L_\varphi = L_{\varphi_\lambda}$ .*

*Proof.* The statements follow from  $\lambda U_{\varphi\lambda} = U_{\varphi}$ .  $\square$

**Remark 3.3.** *The results of Lemmas 3.1 and 3.2 can also be obtained via Dickson matrices. For a  $q$ -polynomial  $f$  let  $D_f$  denote the Dickson matrix associated with  $f$ . When  $f(x) = \lambda x$  for some  $\lambda \in \mathbb{F}_{q^n}$  we will simply write  $D_\lambda$ . We will denote the point  $\langle(1, \lambda)\rangle_{q^n}$  by  $P_\lambda$ .*

*Transposition preserves the rank of matrices and  $D_f^T = D_{\hat{f}}$ ,  $D_\lambda^T = D_\lambda$ . It follows that*

$$\dim_q \ker(D_f - D_\lambda) = \dim_q \ker(D_f - D_\lambda)^T = \dim_q \ker(D_{\hat{f}} - D_\lambda),$$

*and hence for each  $\lambda \in \mathbb{F}_{q^n}$  we have  $w_{L_f}(P_\lambda) = w_{L_{\hat{f}}}(P_\lambda)$ .*

*Let  $f_\mu(x) = f(x\mu)/\mu$ . It is easy to see that  $D_{1/\mu}D_fD_\mu = D_{f_\mu}$  and*

$$\dim_q \ker(D_f - D_\lambda) = \dim_q \ker D_{1/\mu}(D_f - D_\lambda)D_\mu = \dim_q \ker(D_{f_\mu} - D_\lambda),$$

*and hence  $w_{L_f}(P_\lambda) = w_{L_{f_\mu}}(P_\lambda)$  for each  $\lambda \in \mathbb{F}_{q^n}$ .*

From the previous arguments it follows that linear sets  $L_f$  with  $f(x) = \hat{f}(x)$  are good candidates for being simple. In the next section we show that the trace function, which has the previous property, defines a simple linear set. We are going to use the following lemmas which will also be useful later.

**Lemma 3.4.** *Let  $f$  and  $g$  be two linearized polynomials. If  $L_f = L_g$ , then for each positive integer  $d$  the following holds*

$$\sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{f(x)}{x} \right)^d = \sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{g(x)}{x} \right)^d.$$

*Proof.* If  $L_f = L_g =: L$ , then  $\{f(x)/x : x \in \mathbb{F}_{q^n}^*\} = \{g(x)/x : x \in \mathbb{F}_{q^n}^*\} =: H$ . For each  $h \in H$  we have  $|\{x : f(x)/x = h\}| = q^i - 1$ , where  $i$  is the weight of the point  $\langle(1, h)\rangle_{q^n} \in L$  w.r.t.  $U_f$ , and similarly  $|\{x : g(x)/x = h\}| = q^j - 1$ , where  $j$  is the weight of the point  $\langle(1, h)\rangle_{q^n} \in L$  w.r.t.  $U_g$ . Because of the characteristic of  $\mathbb{F}_{q^n}$ , we obtain:

$$\sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{f(x)}{x} \right)^d = - \sum_{h \in H} h^d = \sum_{x \in \mathbb{F}_{q^n}^*} \left( \frac{g(x)}{x} \right)^d.$$

$\square$

**Lemma 3.5** (Folklore). *For any prime power  $q$  and integer  $d$  we have  $\sum_{x \in \mathbb{F}_q^*} x^d = -1$  if  $q-1 \mid d$  and  $\sum_{x \in \mathbb{F}_q^*} x^d = 0$  otherwise.*



**Lemma 3.6.** Let  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$  and  $g(x) = \sum_{i=0}^{n-1} b_i x^{q^i}$  be two  $q$ -polynomials over  $\mathbb{F}_{q^n}$ , such that  $L_f = L_g$ . Then

$$a_0 = b_0, \quad (6)$$

and for  $k = 1, 2, \dots, n-1$  it holds that

$$a_k a_{n-k}^{q^k} = b_k b_{n-k}^{q^k}, \quad (7)$$

for  $k = 2, 3, \dots, n-1$  it holds that

$$a_1 a_{k-1}^q a_{n-k}^{q^k} + a_k a_{n-1}^q a_{n-k+1}^{q^k} = b_1 b_{k-1}^q b_{n-k}^{q^k} + b_k b_{n-1}^q b_{n-k+1}^{q^k}. \quad (8)$$

*Proof.* We are going to use Lemma 3.5 together with Lemma 3.4 with different choices of  $d$ .

With  $d = 1$  we have

$$\sum_{x \in \mathbb{F}_{q^n}^*} \sum_{i=0}^{n-1} a_i x^{q^i-1} = \sum_{x \in \mathbb{F}_{q^n}^*} \sum_{i=0}^{n-1} b_i x^{q^i-1},$$

and hence

$$\sum_{i=0}^{n-1} a_i \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i-1} = \sum_{i=0}^{n-1} b_i \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i-1}.$$

Since  $q^n - 1$  cannot divide  $q^i - 1$  with  $i = 1, 2, \dots, n-1$ ,  $a_0 = b_0 =: c$  follows. Let  $\varphi$  denotes the  $\mathbb{F}_q$ -linear map which fixes  $(0, 1)$  and maps  $(1, 0)$  to  $(1, -c)$ . Then  $U_f^\varphi = U_{f'}$  and  $U_g^\varphi = U_{g'}$  with  $f' = \sum_{i=1}^{n-1} a_i x^{q^i}$ ,  $g' = \sum_{i=1}^{n-1} b_i x^{q^i}$  and of course with  $L_{f'} = L_{g'}$ . It follows that we may assume  $c = 0$ .

First we show that (7) holds. With  $d = q^k + 1$ ,  $1 \leq k \leq n-1$  we obtain

$$\sum_{1 \leq i, j \leq n-1} a_i a_j^{q^k} \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i-1+q^{j+k}-q^k} = \sum_{1 \leq i, j \leq n-1} b_i b_j^{q^k} \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i-1+q^{j+k}-q^k}.$$

$\sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i-1+q^{j+k}-q^k} = -1$  if and only if  $q^i + q^{j+k} \equiv q^k + 1 \pmod{q^n - 1}$ , and zero otherwise. Suppose that the former case holds.

First consider  $j + k \leq n-1$ . Then  $q^i + q^{j+k} \leq q^{n-1} + q^{n-1} < q^k + 1 + 2(q^n - 1)$  hence one of the following holds.

- If  $q^i + q^{j+k} = q^k + 1$ , then the right hand side is not divisible by  $q$ , a contradiction.
- If  $q^i + q^{j+k} = q^k + 1 + (q^n - 1) = q^n + q^k$ , then  $j + k = n$ , a contradiction.

Now consider the case  $j + k \geq n$ . Then  $q^i + q^{j+k} \equiv q^i + q^{j+k-n} \equiv q^k + 1 \pmod{q^n - 1}$ . Since  $j + k \leq 2(n - 1)$ , we have  $q^i + q^{j+k-n} \leq q^{n-1} + q^{n-2} < q^k + 1 + 2(q^n - 1)$ , hence one of the following holds.

- If  $q^i + q^{j+k-n} = q^k + 1$ , then  $j + k = n$  and  $i = k$ .
- If  $q^i + q^{j+k-n} = q^k + 1 + (q^n - 1) = q^n + q^k$ , then there is no solution since  $j + k - n \notin \{k, n\}$ .

Hence (7) follows. Now we show that (8) also holds. Note that in this case  $n \geq 3$ , otherwise there is no  $k$  with  $2 \leq k \leq n - 1$ . With  $d = q^k + q + 1$ , we obtain

$$\sum_{1 \leq i, j, m \leq n-1} a_i a_j^q a_m^{q^k} \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+k} - q^k} = \sum_{1 \leq i, j, m \leq n-1} b_i b_j^q b_m^{q^k} \sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+k} - q^k}.$$

$\sum_{x \in \mathbb{F}_{q^n}^*} x^{q^i - 1 + q^{j+1} - q + q^{m+k} - q^k} = -1$  if and only if  $q^i + q^{j+1} + q^{m+k} \equiv q^k + q + 1 \pmod{q^n - 1}$ , and zero otherwise. Suppose that the former case holds.

First consider  $m + k \leq n - 1$ . Then  $q^i + q^{j+1} + q^{m+k} \leq q^{n-1} + q^n + q^{n-1} < q^k + q + 1 + 2(q^n - 1)$  hence one of the following holds.

- If  $q^i + q^{j+1} + q^{m+k} = q^k + q + 1$ , then the right hand side is not divisible by  $q$ , a contradiction.
- If  $q^i + q^{j+1} + q^{m+k} = q^k + q + 1 + (q^n - 1) = q^n + q^k + q$ , then  $m + k = n$ ,  $j + 1 = k$  and  $i = 1$ , a contradiction.

Now consider the case  $m + k \geq n$ . Then  $q^i + q^{j+1} + q^{m+k} \equiv q^i + q^{j+1} + q^{m+k-n} \equiv q^k + q + 1 \pmod{q^n - 1}$ . We have  $q^i + q^{j+1} + q^{m+k-n} \leq q^{n-1} + q^n + q^{n-2} < q^k + q + 1 + 2(q^n - 1)$  hence one of the following holds.

- If  $q^i + q^{j+1} + q^{m+k-n} = q^k + q + 1$ , then  $j + 1 = k$ ,  $i = 1$  and  $m + k = n$ .
- If  $q^i + q^{j+1} + q^{m+k-n} = q^k + q + 1 + (q^n - 1) = q^n + q^k + q$ , then  $j + 1 = n$ ,  $i = k$  and  $m + k = n + 1$ .

This concludes the proof.  $\square$

### 3.1 Linear sets defined by the trace function

We show that there exist at least one simple  $\mathbb{F}_q$ -linear set in  $\text{PG}(1, q^n)$  for each  $q$  and  $n$ . Let  $V = \{(x, \text{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}\}$ . We show that  $L_U = L_V$  occurs for an  $\mathbb{F}_q$ -subspace  $U$  of  $W$  if and only if  $V = \lambda U$  for some  $\lambda \in \mathbb{F}_{q^n}^*$ , i.e.  $L_V$  is of  $\mathcal{Z}(\text{GL})$ -class one. For the special case when  $L_U$  has a point of weight  $n - 1$  see also [7, Theorem 2.3].

**Theorem 3.7.** *The  $\mathbb{F}_q$ -subspace  $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$  defines the same linear set of  $\text{PG}(1, q^n)$  as the  $\mathbb{F}_q$ -subspace  $V = \{(x, \text{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}\}$  if and only if  $\lambda U_f = V$  for some  $\lambda \in \mathbb{F}_{q^n}^*$ , i.e.  $L_V$  is simple.*

*Proof.* Let  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ . We are going to use Lemma 3.6 with  $g(x) = \text{Tr}_{q^n/q}(x)$ . The coefficients  $b_0, b_1, \dots, b_{n-1}$  of  $g(x)$  are 1, hence  $a_0 = 1$ , and for  $k = 1, 2, \dots, n - 1$

$$a_k a_{n-k}^{q^k} = 1, \quad (9)$$

for  $k = 2, 3, \dots, n - 1$

$$a_1 a_{k-1}^q a_{n-k}^{q^k} + a_k a_{n-1}^q a_{n-k+1}^{q^k} = 2. \quad (10)$$

Note that (9) implies  $a_i \neq 0$  for  $i = 1, 2, \dots, n - 1$ . First we prove

$$a_i = a_1^{1+q+\dots+q^{i-1}} \quad (11)$$

by induction on  $i$  for each  $0 < i < n$ . The assertion holds for  $i = 1$ . Suppose that it holds for some integer  $i - 1$  with  $1 < i < n$ . We prove that it also holds for  $i$ . Then (10) with  $k = i$  gives

$$a_1 a_{i-1}^q a_{n-i}^{q^i} + a_i a_{n-1}^q a_{n-i+1}^{q^i} = 2. \quad (12)$$

Also, (9) with  $k = i$ ,  $k = i - 1$  and  $k = 1$ , respectively, gives

$$a_{n-i}^{q^i} = 1/a_i,$$

$$a_{n-i+1}^{q^i} = 1/a_{i-1}^q,$$

$$a_{n-1}^q = 1/a_1.$$

Then (12) gives

$$a_1 a_{i-1}^q / a_i + a_i / (a_1 a_{i-1}^q) = 2. \quad (13)$$

It follows that  $a_1 a_{i-1}^q / a_i = 1$  and hence the induction hypothesis on  $a_{i-1}$  yields  $a_i = a_1^{1+q+\dots+q^{i-1}}$ .

Finally we show  $N(a_1) = 1$ . First consider  $n$  even. Then (9) with  $k = n/2$  gives  $a_{n/2}^{q^{n/2}+1} = 1$ . Applying (11) yields  $N(a_1) = 1$ . If  $n$  is odd, then (9) with  $k = (n-1)/2$  gives  $a_{(n-1)/2} a_{(n+1)/2}^{q^{(n-1)/2}} = 1$ . Applying (11) yields  $N(a_1) = 1$ . It follows that  $a_1 = \lambda^{q-1}$  for some  $\lambda \in \mathbb{F}_{q^n}^*$  and hence  $f(x) = \sum_{i=0}^{n-1} \lambda^{q^i-1} x^{q^i}$ . Then  $\lambda U_f = \{(x, \text{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}^*\}$ .  $\square$

### 3.2 Non-simple linear sets

So far, the only known non-simple linear sets of  $\text{PG}(1, q^n)$  are those of pseudoregulus type when  $n = 5$ , or  $n > 6$ , see Remark 5.6. Now we want to show that  $\mathbb{F}_q$ -linear sets  $L_f$  of  $\text{PG}(1, q^n)$  introduced by Lunardon and Polverino, which are not of pseudoregulus type ([21, Theorems 2 and 3], are non-simple as well. Let start by proving the following preliminary result.

**Proposition 3.8.** *Let  $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ . There is an  $\mathbb{F}_{q^n}$ -semilinear map between  $U_f$  and  $U_{\hat{f}}$  if and only if the following system of  $n$  equations has a solution  $A, B, C, D \in \mathbb{F}_{q^n}$ ,  $AD - BC \neq 0$ ,  $\sigma = p^k$ :*

$$\begin{aligned} C + Da_0^\sigma - a_0 A &= \sum_{i=0}^{n-1} (Ba_i a_i^\sigma)^{q^{n-i}}, \\ &\dots \\ Da_m^\sigma - (a_{n-m} A)^{q^m} &= \sum_{i=0}^{n-1} (Ba_i a_{i+m}^\sigma)^{q^{n-i}}, \\ &\dots \\ Da_{n-1}^\sigma - (a_1 A)^{q^{n-1}} &= \sum_{i=0}^{n-1} (Ba_i a_{i+n-1}^\sigma)^{q^{n-i}}, \end{aligned}$$

where the indices are taken modulo  $n$ .

*Proof.* Because of cardinality reasons the condition  $AD - BC \neq 0$  is necessary. Then

$$\{(x, \hat{f}(x)) : x \in \mathbb{F}_{q^n}\} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x^\sigma \\ f(x)^\sigma \end{pmatrix} : x \in \mathbb{F}_{q^n} \right\}$$

holds if and only if

$$Cx^\sigma + D \sum_{j=0}^{n-1} a_j^\sigma x^{\sigma q^j} = \sum_{i=0}^{n-1} a_{n-i}^{q^i} \left( Ax^\sigma + B \sum_{j=0}^{n-1} a_j^\sigma x^{\sigma q^j} \right)^{q^i}$$

for each  $x \in \mathbb{F}_{q^n}$ . After reducing modulo  $x^{q^n} - x$ , this is a polynomial equation of degree  $q^{n-1}$  in the variable  $x^\sigma$ . It follows that it holds for each  $x \in \mathbb{F}_{q^n}$  if and only if it is the zero polynomial. Comparing coefficients on both sides yields the assertion.  $\square$

We are able to prove the following.

**Proposition 3.9.** *Consider a polynomial of the form  $f(x) = \delta x^q + x^{q^{n-1}}$ , where  $q > 4$  is a power of the prime  $p$ . If  $n > 4$ , then for each generator  $\delta$  of the multiplicative group of  $\mathbb{F}_{q^n}$  the linear set  $L_f$  is not simple.*

*Proof.* Lemma 3.1 yields  $L_f = L_{\hat{f}}$  thus it is enough to show the existence of  $\delta$  such that there is no  $\mathbb{F}_{q^n}$ -semilinear map between  $U_f$  and  $U_{\hat{f}}$ . In the equations of Proposition 3.8 we have  $a_1 = \delta$ ,  $a_{n-1} = 1$  and  $a_0 = a_2 = \dots = a_{n-2} = 0$ , thus

$$\begin{aligned} C &= (B\delta^{\sigma+1})^{q^{n-1}} + B^q, \\ D\delta^\sigma - A^q &= 0, \\ 0 &= (B\delta)^{q^{n-1}}, \\ D - (\delta A)^{q^{n-1}} &= 0, \end{aligned}$$

where  $\sigma = p^k$  for some integer  $k$ . If there is a solution, then  $B = C = 0$  and  $(\delta A)^{q^{n-1}} \delta^\sigma = A^q$ . Taking  $q$ -th powers on both sides yield

$$\delta^{\sigma q+1} = A^{q^2-1} \tag{14}$$

and hence

$$\delta^{\frac{(\sigma q+1)(q^n-1)}{q-1}} = 1. \tag{15}$$

For each  $\sigma$  let  $G_\sigma$  be the set of elements  $\delta$  of  $\mathbb{F}_{q^n}$  satisfying (15). For each  $\sigma$ ,  $G_\sigma$  is a subgroup of the multiplicative group  $M$  of  $\mathbb{F}_{q^n}$ . We show that these are proper subgroups of  $M$ . We have  $G_{p^k} = M$  if and only if  $q^n - 1$  divides  $\frac{(p^k q+1)(q^n-1)}{q-1}$ , i.e. when  $q-1$  divides  $p^k q + 1$ . Since  $\gcd(p^w + 1, p^v - 1)$  is always 1, 2, or  $p^{\gcd(w,v)} + 1$ , it follows that for  $q > 4$  we cannot have  $q-1$  as a divisor of  $p^k q + 1$ .

It follows that for any generator  $\delta$  of  $M$  we have  $\delta \notin \cup_j G_{p^j}$  and hence  $\delta^{\sigma q+1} \neq A^{q^2-1}$  for each  $\sigma$  and for each  $A$ .  $\square$

**Remark 3.10.** *If  $q = 4$ , then (14) with  $k = 2(n-1)+1$  asks for the solution of  $\delta^3 = A^{15}$ . When 5 does not divide  $4^n - 1$ , then  $\{x^3 : x \in \mathbb{F}_{4^n}\} = \{x^{15} : x \in \mathbb{F}_{4^n}\}$  and hence for each  $\delta$  there exists  $A$  such that  $\delta^3 = A^{15}$ .*

If  $q = 3$ , then (14) with  $k = n - 1$  asks for the solution of  $\delta^2 = A^8$ . When 4 does not divide  $3^n - 1$ , then  $\{x^2 : x \in \mathbb{F}_{3^n}\} = \{x^8 : x \in \mathbb{F}_{3^n}\}$  and hence for each  $\delta$  there exists  $A$  such that  $\delta^2 = A^8$ .

If  $q = 2$ , then (14) with  $k = 0$  asks for the solution of  $\delta^3 = A^3$ . This equation always has a solution.

## 4 Linear sets of rank 4 of $\text{PG}(1, q^4)$

$\mathbb{F}_q$ -linear sets of rank two of  $\text{PG}(1, q^2)$  are the Baer sublines, which are equivalent. As we have mentioned in the introduction, subgeometries are simple linear sets, in fact they have  $\mathcal{Z}(\Gamma\text{L})$ -class one (cf. [17, Theorem 2.6] and [13, Section 25.5]). There are two non-equivalent  $\mathbb{F}_q$ -linear sets of rank 3 of  $\text{PG}(1, q^3)$ , the linear sets of size  $q^2 + q + 1$  and those of size  $q^2 + 1$ . Linear sets in both families are equivalent, since the stabilizer of a  $q$ -order subgeometry  $\Sigma$  of  $\Sigma^* = \text{PG}(2, q^3)$  is transitive on the set of those points of  $\Sigma^* \setminus \Sigma$  which are incident with a line of  $\Sigma$  and on the set of points of  $\Sigma^*$  not incident with any line of  $\Sigma$  (cf. Section 5.2 and [16]). In the first case we have the linear sets of pseudoregulus type with  $\Gamma\text{L}$ -class 1 and  $\mathcal{Z}(\Gamma\text{L})$ -class 2 (cf. Remark 5.6 and Example 5.1). In the second case we have the linear sets defined by  $\text{Tr}_{q^3/q}$  with  $\Gamma\text{L}$ -class and  $\mathcal{Z}(\Gamma\text{L})$ -class 1 (cf. Theorem 3.7, see also [11, Corollary 6]).

The main result of this section is that each  $\mathbb{F}_q$ -linear set of rank 4 of  $\text{PG}(1, q^4)$ , with maximum field of linearity  $\mathbb{F}_q$ , is simple (cf. Theorem 4.5).

### 4.1 Subspaces defining the same linear set

**Lemma 4.1.** *Let  $f(x) = \sum_{i=0}^3 a_i x^{q^i}$  and  $g(x) = \sum_{i=0}^3 b_i x^{q^i}$  be two  $q$ -polynomials over  $\mathbb{F}_{q^4}$ , such that  $L_f = L_g$ . Then*

$$\begin{aligned} N(a_1) + N(a_2) + N(a_3) + a_1^{1+q^2} a_3^{q+q^3} + a_1^{q+q^3} a_3^{1+q^2} + \text{Tr } q^4/q \left( a_1 a_2^{q+q^2} a_3^{q^3} \right) = \\ N(b_1) + N(b_2) + N(b_3) + b_1^{1+q^2} b_3^{q+q^3} + b_1^{q+q^3} b_3^{1+q^2} + \text{Tr } q^4/q \left( b_1 b_2^{q+q^2} b_3^{q^3} \right). \end{aligned}$$

*Proof.* We are going to follow the proof of Lemma 3.6. As in that proof, we may assume  $a_0 = b_0 = 0$ . In Lemma 3.4 take  $d = 1 + q + q^2 + q^3$ . We obtain

$$\begin{aligned} \sum_{1 \leq i, j, k, m \leq 3} a_i a_j^q a_k^{q^2} a_m^{q^3} \sum_{x \in \mathbb{F}_{q^4}^*} x^{q^i - 1 + q^{j+1} - q + q^{k+2} - q^2 + q^{m+3} - q^3} = \\ \sum_{1 \leq i, j, k, m \leq 3} b_i b_j^q b_k^{q^2} b_m^{q^3} \sum_{x \in \mathbb{F}_{q^4}^*} x^{q^i - 1 + q^{j+1} - q + q^{k+2} - q^2 + q^{m+3} - q^3}. \end{aligned}$$

$\sum_{x \in \mathbb{F}_{q^4}^*} x^{q^i-1+q^{j+1}-q+q^{k+2}-q^2+q^{m+3}-q^3} = -1$  if and only if

$$q^i + q^{j+1} + q^{k+2} + q^{m+3} \equiv q^i + q^{j+1} + q^{k+2} + q^{m-1} \equiv 1 + q + q^2 + q^3 \pmod{q^4-1},$$

and zero otherwise. Suppose that the former case holds.

First consider  $k = 1$ . Then  $q^i + q^{j+1} + q^{k+2} + q^{m-1} \leq q^3 + q^4 + q^3 + q^2 < 1 + q + q^2 + q^3 + 2(q^4 - 1)$  hence one of the following holds.

- If  $q^i + q^{j+1} + q^{k+2} + q^{m-1} = 1 + q + q^2 + q^3$ , then  $m = i = j = k = 1$ .
- If  $q^i + q^{j+1} + q^{k+2} + q^{m-1} = 1 + q + q^2 + q^3 + q^4 - 1 = q + q^2 + q^3 + q^4$ , then  $\{i, j+1, k+2, m-1\} = \{1, 2, 3, 4\}$ , hence one of the following holds

$$i = 1, j = 3, k = 1, m = 3,$$

$$i = 2, j = 3, k = 1, m = 2.$$

Now consider the case  $k \geq 2$ . Then  $q^i + q^{j+1} + q^{k+2} + q^{m-1} \equiv q^i + q^{j+1} + q^{k-2} + q^{m-1} \leq q^3 + q^4 + q + q^2 < 1 + q + q^2 + q^3 + 2(q^4 - 1)$  hence one of the following holds.

- If  $q^i + q^{j+1} + q^{k-2} + q^{m-1} = 1 + q + q^2 + q^3$ , then  $\{i, j+1, k-2, m-1\} = \{0, 1, 2, 3\}$ , hence one of the following holds

$$i = 1, j = 2, k = 2, m = 3,$$

$$i = 2, j = 2, k = 2, m = 2,$$

$$i = 2, j = 2, k = 3, m = 1,$$

$$i = 3, j = 1, k = 2, m = 2,$$

$$i = 3, j = 1, k = 3, m = 1.$$

- If  $q^i + q^{j+1} + q^{k-2} + q^{m-1} = 1 + q + q^2 + q^3 + q^4 - 1 = q + q^2 + q^3 + q^4$ , then  $i = j = k = m = 3$ .

□

**Proposition 4.2.** *Let  $f(x)$  and  $g(x)$  be two  $q$ -polynomials over  $\mathbb{F}_{q^4}$  such that  $L_f = L_g$ . If the maximum field of linearity of  $f$  is  $\mathbb{F}_q$ , then*

$$g(x) = f(\lambda x)/\lambda,$$

or

$$g(x) = \hat{f}(\lambda x)/\lambda.$$

*Proof.* By Proposition 2.2, the maximum field of linearity of  $g$  is also  $\mathbb{F}_q$ . First note that  $L_g = L_f$  when  $g$  is as in the assertion (cf. Lemmas 3.1 and 3.2). Let  $f(x) = \sum_{i=0}^3 a_i x^{q^i}$  and  $g(x) = \sum_{i=0}^3 b_i x^{q^i}$ .

First we are going to use Lemma 3.6. From (6) we have  $a_0 = b_0$ . From (7) with  $n = 4$  and  $k = 1, 2$  we have  $a_1 a_3^q = b_1 b_3^q$  and  $a_2^{1+q^2} = b_2^{1+q^2}$ , respectively. From (8) with  $n = 4$  and  $k = 2$  we obtain

$$a_1^{q+1} a_2^{q^2} + a_2 a_3^{q+q^2} = b_1^{q+1} b_2^{q^2} + b_2 b_3^{q+q^2}. \quad (16)$$

Note that  $a_1 a_3^q = b_1 b_3^q$  implies

$$N(b_1) N(b_3) = N(a_1) N(a_3). \quad (17)$$

Multiplying (16) by  $b_2$  and applying  $a_2^{1+q^2} = b_2^{1+q^2}$  yields:

$$b_2^2 b_3^{q^2+q} - b_2 (a_1^{q+1} a_2^{q^2} + a_2 a_3^{q^2+q}) + b_1^{q+1} a_2^{q^2+1} = 0. \quad (18)$$

First suppose  $b_1 b_2 b_3 \neq 0$ . Then (18) is a second degree polynomial in  $b_2$ . Applying  $a_1 a_3^q = b_1 b_3^q$  it is easy to see that the roots of (18) are

$$b_{2,1} = \frac{a_1^{q+1} a_2^{q^2}}{b_3^{q^2+q}},$$

$$b_{2,2} = \frac{a_2 a_3^{q^2+q}}{b_3^{q^2+q}}.$$

First we consider  $b_2 = b_{2,1}$ . Then  $a_2^{1+q^2} = b_2^{1+q^2}$  yields  $N(a_1) = N(b_3)$  and hence  $N(b_1) = N(a_3)$ . In particular,  $N(b_1/a_3^q) = 1$  and hence  $b_1 = a_3^q \lambda^{q-1}$  for some  $\lambda \in \mathbb{F}_{q^4}^*$ . From  $a_1 a_3^q = b_1 b_3^q$  we obtain  $b_3 = a_1^{q^3} a_3 / b_1^{q^3} = a_1^{q^3} \lambda^{q^3-1}$ . Applying this we get  $b_2 = a_1^{q+1} a_2^{q^2} / b_3^{q^2+q} = a_2^{q^2} \lambda^{q^2-1}$  and hence

$$g(x) = a_0 x + a_3^q \lambda^{q-1} x^q + a_2^{q^2} \lambda^{q^2-1} x^{q^2} + a_1^{q^3} \lambda^{q^3-1} x^{q^3}.$$

as we claimed.

Now consider  $b_2 = b_{2,2}$ . Then  $a_2^{1+q^2} = b_2^{1+q^2}$  yields  $N(a_3) = N(b_3)$  and hence  $N(a_1) = N(b_1)$ . Hence  $b_1 = a_1 \lambda^{q-1}$  for some  $\lambda \in \mathbb{F}_{q^4}^*$ . From  $a_1 a_3^q = b_1 b_3^q$  we obtain  $b_3 = a_1^{q^3} a_3 / b_1^{q^3} = a_3 \lambda^{q^3-1}$ . Applying this we obtain  $b_2 = a_2 a_3^{q^2+q} / b_3^{q^2+q} = a_2 \lambda^{q^2-1}$  and hence

$$g(x) = a_0 x + a_1 \lambda^{q-1} x^q + a_2^{q^2} \lambda^{q^2-1} x^{q^2} + a_3^{q^3} \lambda^{q^3-1} x^{q^3}.$$



If  $b_1 = b_3 = 0$ , then either  $b_2 = 0$  and the maximum field of linearity of  $g(x)$  is  $\mathbb{F}_{q^4}$ , or  $b_2 \neq 0$  and the maximum field of linearity of  $g(x)$  is  $\mathbb{F}_{q^2}$ . Thus we may assume  $b_1 \neq 0$  or  $b_3 \neq 0$ .

First assume  $b_2 \neq 0$  and  $b_1 = 0$ . Then  $b_3 \neq 0$  and (18) gives

$$b_2 b_3^{q^2+q} = a_1^{q+1} a_2^{q^2} + a_2 a_3^{q^2+q}.$$

Then  $a_1 a_3^q = b_1 b_3^q$  yields either  $a_1 = 0$  and  $b_2 b_3^{q^2+q} = a_2 a_3^{q^2+q}$ , or  $a_3 = 0$  and  $b_2 b_3^{q^2+q} = a_1^{q+1} a_2^{q^2}$ . Taking  $(q^2 + 1)$ -powers on both sides gives  $b_2^{q^2+1} N(b_3) = a_2^{q^2+1} N(a_3)$ , or  $b_2^{q^2+1} N(b_3) = N(a_1) a_2^{q^2+1}$ , respectively. Applying  $b_2^{q^2+1} = a_2^{q^2+1}$  we get  $N(b_3) = N(a_3)$ , or  $N(b_3) = N(a_1)$ , respectively. Note that the set of elements with norm 1 in  $\mathbb{F}_{q^4}$  is  $\{x^{q^3-1} : x \in \mathbb{F}_{q^4}^*\}$ , thus in the first case there exists  $\lambda \in \mathbb{F}_{q^4}^*$  such that  $b_3 = a_3 \lambda^{q^3-1}$ . Then  $b_2 b_3^{q^2+q} = a_2 a_3^{q^2+q}$  yields  $b_2 = a_2 \lambda^{q^2-1}$  and hence  $g(x) = a_0 x + a_2 \lambda^{q^2-1} x^{q^2} + a_3 \lambda^{q^3-1} x^{q^3}$ . In the second case the same reasoning yields  $g(x) = a_0 x + a_2^{q^2} \lambda^{q^2-1} x^{q^2} + a_1^{q^3} \lambda^{q^3-1} x^{q^3}$ .

If  $b_2 \neq 0$  and  $b_3 = 0$ , then the coefficient of  $x^q$  in  $\hat{g}(x)$  is zero and the assertion follows from the above arguments applied to  $\hat{g}$  instead of  $g$ .

Now assume  $b_2 = 0$  and  $b_1 b_3 = 0$ . Then  $L_g = L_f$  is a linear set of pseudoregulus type and hence the assertion also follows from [15]. For the sake of completeness we present a proof also in this case. Equation  $b_2^{q^2+1} = a_2^{q^2+1}$  yields  $a_2 = 0$  and equation  $a_1 a_3^q = b_1 b_3^q$  yields  $a_1 a_3 = 0$ . Then from Lemma 4.1 we have

$$N(a_1) + N(a_3) = N(b_1) + N(b_3). \quad (19)$$

If  $b_1 = 0$ , then  $b_3 \neq 0$  and either  $a_1 = 0$  and  $N(a_3) = N(b_3)$ , or  $a_3 = 0$  and  $N(a_1) = N(b_3)$ . In the first case  $g(x) = a_0 x + a_3 \lambda^{q^3-1} x^{q^3}$ , in the second case  $g(x) = a_0 x + a_1^q \lambda^{q^3-1} x^{q^3}$ . If  $b_3 = 0$ , then  $b_1 \neq 0$  and either  $a_1 = 0$  and  $N(a_3) = N(b_1)$ , or  $a_3 = 0$  and  $N(a_1) = N(b_1)$ . In the first case  $g(x) = a_0 x + a_3^q \lambda^{q^3-1} x^{q^3}$ , in the second case  $g(x) = a_0 x + a_1 \lambda^{q^3-1} x^{q^3}$ .

There is only one case left, when  $b_2 = 0$  and  $b_1 b_3 \neq 0$ . Then from Lemma 4.1 and from  $a_1 a_3^q = b_1 b_3^q$  it follows that

$$N(a_1) + N(a_3) = N(b_1) + N(b_3). \quad (20)$$

Together with (17) it follows that either  $N(a_1) = N(b_1)$  and  $N(a_3) = N(b_3)$ , or  $N(a_1) = N(b_3)$  and  $N(a_3) = N(b_1)$ . In the first case  $g(x) = a_0 x + a_1 \lambda^{q-1} x^q + a_3 \lambda^{q^3-1} x^{q^3}$ , in the second case  $g(x) = a_0 x + a_3^q \lambda^{q-1} x^q + a_1^{q^3} \lambda^{q^3-1} x^{q^3}$ , for some  $\lambda \in \mathbb{F}_{q^4}^*$ .  $\square$

Now we are able to prove the following.

**Theorem 4.3.** *Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of a line  $\text{PG}(W, \mathbb{F}_{q^4})$  of rank 4, with maximum field of linearity  $\mathbb{F}_q$ , and let  $\beta$  be a non-degenerate alternating form of  $W$ . If  $V$  is an  $\mathbb{F}_q$ -vector subspace of  $W$  such that  $L_U = L_V$ , then either*

$$V = \mu U,$$

or

$$V = \mu U^{\perp'_\beta},$$

for some  $\mu \in \mathbb{F}_{q^4}^*$ , where  $\perp'_\beta$  is the orthogonal complement map induced by  $\text{Tr}_{q^4/q} \circ \beta$  on the lattice of the  $\mathbb{F}_q$ -subspaces of  $W$ .

*Proof.* First of all, observe that if  $\beta_1$  is another non-degenerate alternating form of  $W$  and  $\perp'_{\beta_1}$  is the corresponding orthogonal complement map induced on the lattice the  $\mathbb{F}_q$ -subspaces of  $W$ , direct computations show that there exists  $a \in \mathbb{F}_{q^4}^*$  such that  $\beta_1 = a\beta$  and for each  $\mathbb{F}_q$ -vector subspace  $S$  of  $W$  we get  $S^{\perp'_{\beta_1}} = aS^{\perp'_\beta}$ .

Let  $\phi$  be the collineation of  $\text{PG}(W, \mathbb{F}_{q^4})$  such that  $L_U^\phi$  does not contain the point  $\langle(0, 1)\rangle_{\mathbb{F}_{q^4}}$ . Then  $L_{U^\phi} = L_{V^\phi}$ , where  $\phi$  is the invertible  $\mathbb{F}_{q^4}$ -semilinear map of  $W$  inducing  $\phi$ , and  $\sigma$  is the associated field automorphism. Also,  $U^\phi = U_f$  and  $V^\phi = V_g$  for two  $q$ -polynomials  $f$  and  $g$  over  $\mathbb{F}_{q^4}$ . Since  $L_f = L_g$ , by Proposition 4.2 and by Lemma 3.2, taking also (5) into account, it follows that there exists  $\lambda \in \mathbb{F}_{q^4}^*$  such that either  $\lambda V_g = U_f$  or  $\lambda V_g = U_{\hat{f}} = U_f^{\perp'}$ , where  $\perp'$  is the orthogonal complement map induced by the non-degenerate alternating form defined in (4). In the first case we have that  $V = \mu U$ , where  $\mu = \frac{1}{\lambda^{\sigma-1}}$ . In the second case we have  $V = \frac{1}{\lambda^{\sigma-1}} U^\phi \perp' \varphi^{-1}$ . The map  $\varphi \perp' \varphi^{-1}$  defines the orthogonal complement map on the lattice the  $\mathbb{F}_q$ -subspaces of  $W$  induced by another non-degenerate alternating form of  $W$ . As observed above, there exists  $a \in \mathbb{F}_{q^4}^*$  such that  $U^\phi \perp' \varphi^{-1} = aU^{\perp'_\beta}$ . The assertion follows with  $\mu = \frac{a}{\lambda^{\sigma-1}}$ .  $\square$

## 4.2 Semilinear maps between $U_f$ and $U_{\hat{f}}$

The next result is just Proposition 3.8 with  $n = 4$ .

**Corollary 4.4.** *Let  $f(x) = a_0x + a_1x^q + a_2x^{q^2} + a_3x^{q^3}$ . There is an  $\mathbb{F}_{q^4}$ -semilinear map between  $U_f$  and  $U_{\hat{f}}$  if and only if the following system of four equations has a solution  $A, B, C, D \in \mathbb{F}_{q^4}$ ,  $AD - BC \neq 0$ ,  $\sigma = p^k$ .*

$$C + Da_0^\sigma - a_0A = Ba_0a_0^\sigma + (Ba_1a_1^\sigma)^{q^3} + (Ba_2a_2^\sigma)^{q^2} + (Ba_3a_3^\sigma)^q,$$

$$\begin{aligned}
Da_1^\sigma - (a_3A)^q &= Ba_0a_1^\sigma + (Ba_1a_2^\sigma)^{q^3} + (Ba_2a_3^\sigma)^{q^2} + (Ba_3a_0^\sigma)^q, \\
Da_2^\sigma - (a_2A)^{q^2} &= Ba_0a_2^\sigma + (Ba_1a_3^\sigma)^{q^3} + (Ba_2a_0^\sigma)^{q^2} + (Ba_3a_1^\sigma)^q, \\
Da_3^\sigma - (a_1A)^{q^3} &= Ba_0a_3^\sigma + (Ba_1a_0^\sigma)^{q^3} + (Ba_2a_1^\sigma)^{q^2} + (Ba_3a_2^\sigma)^q.
\end{aligned}$$

**Theorem 4.5.** *Linear sets of rank 4 of  $\text{PG}(1, q^4)$ , with maximum field of linearity  $\mathbb{F}_q$ , are simple.*

*Proof.* Let  $f = \sum_{i=0}^3 a_i x^{q^i}$ . After a suitable projectivity we may assume  $a_0 = 0$ . We will use Corollary 4.4 with  $\sigma \in \{1, q^2\}$ . We may assume that  $a_1 = 0$  and  $a_3 = 0$  do not hold at the same time since otherwise  $f$  is  $\mathbb{F}_{q^2}$ -linear.

First consider the case when  $N(a_1) = N(a_3)$ . Let  $B = C = 0$ ,  $D = A^{q^2}$  and take  $A$  such that  $A^{q-1} = a_3/a_1^q$ . This can be done since  $N(a_3/a_1^q) = 1$ . Then Corollary 4.4 with  $\sigma = q^2$  provides the existence of an  $\mathbb{F}_{q^4}$ -semilinear map between  $U_f$  and  $U_{\hat{f}}$ .

From now on we assume  $N(a_1) \neq N(a_3)$ .

If  $a_2 = a_1 = 0$ , then let  $\sigma = 1$ ,  $A = D = 0$ ,  $B = 1$  and  $C = a_3^{2q}$ . If  $a_2 = a_3 = 0$ , then let  $\sigma = 1$ ,  $A = D = 0$ ,  $B = 1$  and  $C = a_1^{2q^3}$ .

Now consider the case  $a_2 = 0$  and  $a_1a_3 \neq 0$ . Let  $A = D = 0$ . Then the equations of Corollary 4.4 with  $\sigma = 1$  yield

$$C = B^{q^3} a_1^{2q^3} + B^q a_3^{2q}, \quad (21)$$

$$0 = B^q a_1^q a_3^q + B^{q^3} a_1^{q^3} a_3^{q^3}. \quad (22)$$

(22) is equivalent to  $0 = (Ba_1a_3)^{q^2} + Ba_1a_3$ . Since  $X^{q^2} + X = 0$  has  $q^2$  solutions in  $\mathbb{F}_{q^4}$ , for any  $a_1$  and  $a_3$  we can find  $B \in \mathbb{F}_{q^4}^*$  such that (22) is satisfied. If  $B^{q^3} a_1^{2q^3} + B^q a_3^{2q} \neq 0$ , then let  $C$  be this field element. We show that this is always the case. Suppose, contrary to our claim, that  $B^{q^3-q} = -a_3^{2q}/a_1^{2q^3}$ . Because of the choice of  $B$  (22) yields  $B^{q^3-q} = -a_1^{q-q^3} a_3^{q-q^3}$ . Since  $B \neq 0$  this implies

$$-a_3^{2q}/a_1^{2q^3} = -a_1^{q-q^3} a_3^{q-q^3},$$

and hence  $a_1^{q^2+1} = a_3^{q^2+1}$ . A contradiction since  $N(a_1) \neq N(a_3)$ . From now on we assume  $a_2 \neq 0$ , we may also assume  $a_2 = 1$  after a suitable projectivity.

Corollary 4.4 with  $\sigma = 1$  yields

$$C = (Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q, \quad (23)$$

$$Da_1 - (a_3A)^q = (Ba_1)^{q^3} + (Ba_3)^{q^2}, \quad (24)$$

$$D - A^{q^2} = (Ba_1a_3)^{q^3} + (Ba_3a_1)^q, \quad (25)$$

$$Da_3 - (a_1A)^{q^3} = (Ba_1)^{q^2} + (Ba_3)^q. \quad (26)$$

The right hand side of (24) is the  $q$ -th power of the right hand side of (26) and hence  $D^qa_3^q - a_1A = Da_1 - a_3^qA^q$ , i.e.

$$a_3^q(D + A)^q = a_1(D + A).$$

Since  $a_1$  or  $a_3$  is non-zero, we have either  $D = -A$ , or  $(D + A)^{q-1} = a_1/a_3^q$ . The latter case can be excluded since in that case  $N(a_1) = N(a_3)$ . Let  $D = -A$ . Then the left hand side of (24) is  $w(A) := -Aa_1 - a_3^qA^q$ . The kernel of  $w$  is trivial and hence  $B$  uniquely determines  $A$ . The inverse of  $w$  is

$$w^{-1}(x) = \frac{-xa_1^{q+q^2+q^3} + x^qa_1^{q^2+q^3}a_3^q - x^{q^2}a_1^{q^3}a_3^{q+q^2} + x^{q^3}a_3^{q+q^2+q^3}}{N(a_1) - N(a_3)}.$$

Denote the right hand side of (24) by  $r(B)$ , the right hand side of (25) by  $t(B)$ . Then  $B$  has to be in the kernel of

$$K(x) := w^{-1}(r(x)) + (w^{-1}(r(x)))^{q^2} + t(x).$$

If  $B = 0$ , then  $A = B = D = 0$  and hence this is not a suitable solution. It is easy to see that  $\text{Im } t \subseteq \mathbb{F}_{q^2}$  and hence also  $\text{Im } K \subseteq \mathbb{F}_{q^2}$ , so the kernel of  $K$  has at least dimension 2.

Let  $B \in \ker K$ ,  $B \neq 0$ ,  $A := w^{-1}(r(B))$  and  $C := (Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q$  (we recall  $D = -A$ ). This gives a solution. We have to check that  $B$  can be chosen such that  $AD - BC \neq 0$ , i.e.

$$Q(B) := (w^{-1}(r(B)))^2 + B \left( (Ba_1^2)^{q^3} + B^{q^2} + (Ba_3^2)^q \right),$$

is non-zero. We have  $w^{-1}(r(x))(N(a_1) - N(a_3)) = \sum_{i=0}^3 c_i x^{q^i}$ , where

$$c_0 = a_1^{1+q^2+q^3}a_3^q - a_1^{q^3}a_3^{1+q+q^2},$$

$$c_1 = a_3^{2q+q^2+q^3} - a_1^{q+q^3}a_3^{q+q^2},$$

$$c_2 = a_3^{q+q^2+q^3}a_1^{q^2} - a_1^{q+q^2+q^3}a_3^{q^2},$$

$$c_3 = a_1^{q^2+q^3}a_3^{q+q^3} - a_1^{q+q^2+2q^3}.$$

If  $X_0, X_1, X_2, X_3$  denote the coordinate functions in  $\text{PG}(3, q^4)$  and  $Q(B) = 0$  for some  $B \in \mathbb{F}_{q^4}$ , then the point  $\langle (B, B^q, B^{q^2}, B^{q^3}) \rangle_{q^4}$  is contained in the quadric  $\mathcal{Q}$  of  $\text{PG}(3, q^4)$  defined by the equation

$$\left( \sum_{i=0}^3 c_i X_i \right)^2 + X_0(X_1 a_3^{2q} + X_2 + X_3 a_1^{2q^3})(N(a_1) - N(a_3))^2 = 0.$$

We can see that the equation of  $\mathcal{Q}$  is the linear combination of the equations of two degenerate quadrics, a quadric of rank 1 and a quadric of rank 2. It follows that  $\mathcal{Q}$  is always singular and it has rank 2 or 3. In particular, the rank of  $\mathcal{Q}$  is 2 when the intersection of the planes  $\mathcal{A} : X_0 = 0$  and  $\mathcal{B} : X_1 a_3^{2q} + X_2 + X_3 a_1^{2q^3} = 0$  is contained in the plane  $\mathcal{C} : \sum_{i=0}^3 c_i X_i = 0$ . Straightforward calculations show that under our hypothesis ( $a_1 \neq 0$  or  $a_3 \neq 0, N(a_1) \neq N(a_3)$ ) this happens if only if  $1 = a_1^q a_3$ .

We recall that the kernel of  $K$  has dimension at least two. Let

$$H = \{ \langle (x, x^q, x^{q^2}, x^{q^3}) \rangle_{q^4} : K(x) = 0 \}.$$

Our aim is to prove that  $H$  has points not belonging to the quadric  $\mathcal{Q}$ , i.e.  $H \not\subseteq \mathcal{Q}$ .

Note that  $x \in \mathbb{F}_{q^4} \mapsto (x, x^q, x^{q^2}, x^{q^3}) \in \mathbb{F}_{q^4}^4$  is a vector-space isomorphism between  $\mathbb{F}_{q^4}$  and the 4-dimensional  $\mathbb{F}_q$ -space  $\{(x, x^q, x^{q^2}, x^{q^3}) : x \in \mathbb{F}_{q^4}\} \subset \mathbb{F}_{q^4}^4$ . Denote by  $\bar{H}$  the  $\mathbb{F}_{q^4}$ -extension of  $H$ , i.e. the projective subspace of  $\text{PG}(3, q^4)$  generated by the points of  $H$ . Then the projective dimension of  $\bar{H}$  is  $\dim \ker K - 1$ . Let  $\sigma$  denotes the collineation  $(X_0, X_1, X_2, X_3) \mapsto (X_3^q, X_0^q, X_1^q, X_2^q)$  of  $\text{PG}(3, q^4)$ . Then the points of  $H$  are fixed points of  $\sigma$  and hence  $\sigma$  fixes the subspace  $\bar{H}$ . Note that the vertex of  $\mathcal{Q}$  is always disjoint from  $H$  since it is contained in  $\mathcal{A}$ , while  $H$  is disjoint from it.

First of all note that if  $\dim \ker K = 4$ , i.e.  $K$  is the zero polynomial, then  $H$  is a subgeometry of  $\text{PG}(3, q^4)$  isomorphic to  $\text{PG}(3, q)$ , which clearly cannot be contained in  $\mathcal{Q}$ . It follows that  $\dim \ker K$  is either 3 or 2, i.e.  $H$  is either a  $q$ -order subplane or a  $q$ -order subline.

First assume  $1 \neq a_1^q a_3$ , i.e. the case when  $\mathcal{Q}$  has rank 3. If  $H$  is a  $q$ -order subplane, then  $H$  cannot be contained in  $\mathcal{Q}$ . To see this, suppose the contrary and take three non-concurrent  $q$ -order sublines of  $H$ . The  $\mathbb{F}_{q^4}$ -extensions of these sublines are also contained in  $\mathcal{Q}$ , but there is at least one of them which does not pass through the singular point of  $\mathcal{Q}$ , a contradiction. Now assume that  $H$  is a  $q$ -order subline. The singular point of  $\mathcal{Q}$  is the intersection of the planes  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$ . Straightforward calculations

show that this point is  $V = \langle (v_0, v_1, v_2, v_3) \rangle_{q^4}$ , where

$$\begin{aligned} v_0 &= 0, \\ v_1 &= a_1^{q^2+q^3} (a_1^{q^3} a_3^{q^2} - 1), \\ v_2 &= a_1^{q^3} a_3^q (a_1^{q^2} a_3^q - a_1^{q^3} a_3^{q^2}), \\ v_3 &= a_3^{q+q^2} (1 - a_1^{q^2} a_3^q). \end{aligned}$$

Suppose, contrary to our claim, that  $H$  is contained in  $\mathcal{Q}$ . Then  $\bar{H}$  passes through the singular point  $V$  of  $\mathcal{Q}$ . Since  $\bar{H}$  is fixed by  $\sigma$ , it follows that the points  $V, V^\sigma, V^{\sigma^2}, V^{\sigma^3}$  have to be collinear ( $v_0 = 0$  yields that these four points cannot coincide). Let  $M$  denote the  $4 \times 4$  matrix, whose  $i$ -th row consists of the coordinates of  $V^{\sigma^{i-1}}$  for  $i = 1, 2, 3, 4$ . The rank of  $M$  is two, thus each of its minors of order three is zero. Let  $M_{i,j}$  denote the submatrix of  $M$  obtained by deleting the  $i$ -th row and  $j$ -th column of  $M$ . Then

$$\begin{aligned} \det M_{1,2} &= a_1^{q+1} (a_1^q a_3 - 1)^{q^3+1} \alpha, \\ \det M_{1,4} &= a_3^{q^3+1} (a_1^q a_3 - 1)^{q^3+1} \beta, \end{aligned}$$

where

$$\begin{aligned} \alpha &= N(a_1)(a_1^{q^2} a_3^q - 1) + N(a_3)(1 - a_1^q a_3 - a_1^{q^3} a_3^{q^2} + a_1 a_3^{q^3}), \\ \beta &= N(a_1)(a_1 a_3^{q^3} + a_1^{q^2} a_3^q - a_1^q a_3 - 1) + N(a_3)(1 - a_1^{q^3} a_3^{q^2}). \end{aligned}$$

Since  $a_1$  and  $a_3$  cannot be both zeros and  $a_1^q a_3 - 1 \neq 0$ , we have  $\alpha = \beta = 0$ . But  $\alpha - \beta = (N(a_1) - N(a_3))(a_1^q a_3 - a_1 a_3^{q^3})$ . It follows that  $a_1^q a_3 \in \mathbb{F}_q$  and hence  $\alpha$  can be written as  $(N(a_1) - N(a_3))(a_1^q a_3 - 1)$ , which is non-zero. This contradiction shows that  $V$  cannot be contained in a line fixed by  $\sigma$  and hence  $\bar{H}$  cannot pass through  $V$ . It follows that  $H \not\subseteq \mathcal{Q}$  and hence we can choose  $B$  such that  $AD - BC \neq 0$ .

Now consider the case  $1 = a_1^q a_3$ . Then  $\mathcal{Q}$  is the union of two planes meeting each other in  $\ell := \mathcal{A} \cap \mathcal{B}$ . It is easy to see that  $R := \langle (0, 1, -a_3^{2q}, 0) \rangle_{q^4}$  and  $R^\sigma$  are two distinct points of  $\ell$ . Since  $N(a_1) \neq N(a_3)$  and  $N(a_1)N(a_3) = 1$ ,  $\det\{R, R^\sigma, R^{\sigma^2}, R^{\sigma^3}\} = N(a_3)^2 - 1$  cannot be zero and hence  $R \notin H$ , otherwise  $\dim\langle R, R^\sigma, R^{\sigma^2}, R^{\sigma^3} \rangle \leq \dim \bar{H} \leq 2$ . Suppose, contrary to our claim, that  $H$  is contained in one of the two planes of  $\mathcal{Q}$ . Since  $R \notin H$ , such a plane can be written as  $\langle H, R \rangle$  and since  $H$  is fixed by  $\sigma$  and  $\ell \subseteq \langle H, R \rangle$ , we have  $\langle H, R \rangle^\sigma = \langle H, R^\sigma \rangle = \langle H, R \rangle$ . Thus  $R, R^\sigma, R^{\sigma^2}, R^{\sigma^3}$  are coplanar, a contradiction.  $\square$

## 5 Different aspects of the classes of a linear set

### 5.1 Class of a linear set and the associated variety

Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $k$  of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(r-1, q^n)$ . Consider the projective space  $\Omega = \text{PG}(W, \mathbb{F}_q) = \text{PG}(rn-1, q)$ . For each point  $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$  of  $\text{PG}(W, \mathbb{F}_{q^n})$  there corresponds a projective  $(n-1)$ -subspace  $X_P := \text{PG}(\langle \mathbf{u} \rangle_{q^n}, \mathbb{F}_q)$  of  $\Omega$ . The variety of  $\Omega$  associated to  $L_U$  is

$$\mathcal{V}_{r,n,k}(L_U) = \bigcup_{P \in L_U} X_P. \quad (27)$$

A  $(k-1)$ -space  $\mathcal{H} = \text{PG}(V, \mathbb{F}_q)$  of  $\Omega$  is said to be a *transversal* space of  $\mathcal{V}(L_U)$  if  $\mathcal{H} \cap X_P \neq \emptyset$  for each point  $P \in L_U$ , i.e.  $L_U = L_V$ .

The  $\mathcal{Z}(\Gamma\text{L})$ -class of an  $\mathbb{F}_q$ -linear set  $L_U$  of rank  $n$  of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$ , with maximum field of linearity  $\mathbb{F}_q$ , is the number of transversal spaces of  $\mathcal{V}_{2,n,n}(L_U)$  up to the action of the subgroup  $G$  of  $\text{PGL}(2n-1, q)$  induced by the maps  $\mathbf{x} \in W \mapsto \lambda \mathbf{x} \in W$ , with  $\lambda \in \mathbb{F}_{q^n}^*$ . Note that  $G$  fixes  $X_P$  for each point  $P \in \text{PG}(1, q^n)$  and hence fixes the variety.

The maximum size of an  $\mathbb{F}_q$ -linear set  $L_U$  of rank  $n$  of  $\text{PG}(1, q^n)$  is  $(q^n - 1)/(q - 1)$ . If this bound is attained (hence each point of  $L_U$  has weight one), then  $L_U$  is a *maximum scattered* linear set of  $\text{PG}(1, q^n)$ . For maximum scattered linear sets, the number of transversal spaces through  $Q \in \mathcal{V}(L_U)$  does not depend on the choice of  $Q$  and this number is the  $\mathcal{Z}(\Gamma\text{L})$ -class of  $L_U$ .

**Example 5.1.** Let  $U = \{(x, x^q) : x \in \mathbb{F}_{q^n}\}$  and consider the linear set  $L_U$ . In [15] the variety  $\mathcal{V}_{2,n,n}(L_U)$  was studied, and the transversal spaces were determined. It follows that the  $\mathcal{Z}(\Gamma\text{L})$ -class of  $L_U$  is  $\varphi(n)$ , where  $\varphi$  is the Euler's phi function.

### 5.2 Classes of linear sets as projections of subgeometries

Let  $\Sigma = \text{PG}(k-1, q)$  be a canonical subgeometry of  $\Sigma^* = \text{PG}(k-1, q^n)$ . Let  $\Gamma \subset \Sigma^* \setminus \Sigma$  be a  $(k-r-1)$ -space and let  $\Lambda \subset \Sigma^* \setminus \Gamma$  be an  $(r-1)$ -space of  $\Sigma^*$ . The projection of  $\Sigma$  from *center*  $\Gamma$  to *axis*  $\Lambda$  is the point set

$$L = p_{\Gamma, \Lambda}(\Sigma) := \{\langle \Gamma, P \rangle \cap \Lambda : P \in \Sigma\}. \quad (28)$$

In [22] Lunardon and Polverino characterized linear sets as projections of canonical subgeometries. They proved the following.

**Theorem 5.2** ([22, Theorems 1 and 2]). *Let  $\Sigma^*$ ,  $\Sigma$ ,  $\Lambda$ ,  $\Gamma$  and  $L = p_{\Gamma, \Lambda}(\Sigma)$  be defined as above. Then  $L$  is an  $\mathbb{F}_q$ -linear set of rank  $k$  and  $\langle L \rangle = \Lambda$ . Conversely, if  $L$  is an  $\mathbb{F}_q$ -linear set of rank  $k$  of  $\Lambda = \text{PG}(r-1, q^n) \subset \Sigma^*$  and  $\langle L \rangle = \Lambda$ , then there is a  $(k-r-1)$ -space  $\Gamma$  disjoint from  $\Lambda$  and a canonical subgeometry  $\Sigma = \text{PG}(r-1, q)$  disjoint from  $\Gamma$  such that  $L = p_{\Gamma, \Lambda}(\Sigma)$ .*

Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $k$  of  $\mathbb{P} = \text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(r-1, q^n)$  such that for each  $k$ -dimensional  $\mathbb{F}_q$ -subspace  $V$  of  $W$  if  $\text{PG}(V, \mathbb{F}_q)$  is a transversal space of  $\mathcal{V}_{r,n,k}(L_U)$ , then there exists  $\gamma \in \text{P}\Gamma\text{L}(W, \mathbb{F}_q)$ , such that  $\gamma$  fixes the Desarguesian spread  $\{X_P : P \in \mathbb{P}\}$  and  $\text{PG}(U, \mathbb{F}_q)^\gamma = \text{PG}(V, \mathbb{F}_q)$ . This is condition (A) from [6], and it is equivalent to say that  $L_U$  is a simple linear set. Then the main results of [6] can be formalized as follows.

**Theorem 5.3** ([6]). *Let  $L_1 = p_{\Gamma_1, \Lambda_1}(\Sigma_1)$  and  $L_2 = p_{\Gamma_2, \Lambda_2}(\Sigma_2)$  be two linear sets of rank  $k$ . If  $L_1$  and  $L_2$  are equivalent and one of them is simple, then there is a collineation mapping  $\Gamma_1$  to  $\Gamma_2$  and  $\Sigma_1$  to  $\Sigma_2$ .*

**Theorem 5.4** ([6]). *If  $L$  is a non-simple linear set of rank  $k$  in  $\Lambda = \langle L \rangle$ , then there are a subspace  $\Gamma = \Gamma_1 = \Gamma_2$  disjoint from  $\Lambda$ , and two  $q$ -order canonical subgeometries  $\Sigma_1, \Sigma_2$  such that  $L = p_{\Gamma, \Lambda}(\Sigma_1) = p_{\Gamma, \Lambda}(\Sigma_2)$ , and there is no collineation fixing  $\Gamma$  and mapping  $\Sigma_1$  to  $\Sigma_2$ .*

Now we interpret the classes of linear sets, hence we are going to consider  $\mathbb{F}_q$ -linear sets of rank  $n$  of  $\Lambda = \text{PG}(1, q^n) = \text{PG}(W, \mathbb{F}_{q^n})$ , with maximum field of linearity  $\mathbb{F}_q$ . Arguing as in the proof of [6, Theorem 7], if  $L_U$  is non-simple, then for any pair  $U, V$  of  $n$ -dimensional  $\mathbb{F}_q$ -subspaces of  $W$  with  $L_U = L_V$  such that  $U^f \neq V$  for each  $f \in \Gamma\text{L}(2, q^n)$  we can find a  $q$ -order subgeometry  $\Sigma$  of  $\Sigma^* = \text{PG}(n-1, q^n)$  and two  $(n-3)$ -spaces  $\Gamma_1$  and  $\Gamma_2$  of  $\Sigma^*$ , disjoint from  $\Sigma$  and from  $\Lambda$ , lying on different orbits of  $\text{Stab}(\Sigma)$ . On the other hand, arguing as in [6, Theorem 6], if there exist two  $(n-3)$ -subspaces  $\Gamma_1$  and  $\Gamma_2$  of  $\Sigma^*$ , disjoint from  $\Sigma$  and from  $\Lambda$ , belonging to different orbits of  $\text{Stab}(\Sigma)$  and such that  $L = p_{\Lambda, \Gamma_1}(\Sigma) = p_{\Lambda, \Gamma_2}(\Sigma)$ , then it is possible to construct two  $n$ -dimensional  $\mathbb{F}_q$ -subspaces  $U$  and  $V$  of  $W$  with  $L_U = L_V$  such that  $U^f \neq V$  for each  $f \in \Gamma\text{L}(2, q^n)$ . Hence we can state the following.

The  $\Gamma\text{L}$ -class of  $L_U$  is the number of orbits of  $\text{Stab}(\Sigma)$  on  $(n-3)$ -spaces of  $\Sigma^*$  containing a  $\Gamma$  disjoint from  $\Sigma$  and from  $\Lambda$  such that  $p_{\Lambda, \Gamma}(\Sigma)$  is equivalent to  $L_U$ .

### 5.3 Class of linear sets and linear blocking sets of Rédei type

A *blocking set*  $\mathcal{B}$  of  $\text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(2, q^n)$  is a point set meeting every line of the plane. Blocking sets of size  $q^n + N \leq 2q^n$  with an  $N$ -secant are called



blocking sets of *Rédei type*, the  $N$ -secants of the blocking set are called *Rédei lines*. Let  $L_U$  be an  $\mathbb{F}_q$ -linear set of rank  $n$  of a line  $\ell = \text{PG}(W, \mathbb{F}_{q^n})$ ,  $W \leq V$ , and let  $\mathbf{w} \in V \setminus W$ . Then  $\langle U, \mathbf{w} \rangle_{\mathbb{F}_q}$  defines an  $\mathbb{F}_q$ -linear blocking set of  $\text{PG}(2, q^n)$  with Rédei line  $\ell$ . The following theorem tells us the number of inequivalent blocking sets obtained in this way.

**Theorem 5.5.** *The  $\Gamma\text{L}$ -class of an  $\mathbb{F}_q$ -linear set  $L_U$  of rank  $n$  of  $\text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$ , with maximum field of linearity  $\mathbb{F}_q$ , is the number of inequivalent  $\mathbb{F}_q$ -linear blocking sets of Rédei type of  $\text{PG}(V, \mathbb{F}_{q^n}) = \text{PG}(2, q^n)$  containing  $L_U$ .*

*Proof.*  $\mathbb{F}_q$ -linear blocking sets of  $\text{PG}(2, q^n)$  with more than one Rédei line are equivalent to those defined by  $\text{Tr}_{q^n/q^m}(x)$  for some divisor  $m$  of  $n$ , see [20, Theorem 5]. Suppose first that  $L_U$  is equivalent to  $L_T$ , where  $T = \{(x, \text{Tr}_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}\}$ . According to Theorem 3.7  $L_T$ , and hence also  $L_U$ , have  $\mathcal{Z}(\Gamma\text{L})$ -class and  $\Gamma\text{L}$ -class one. Proposition 2.5 yields the existence of a unique point  $P \in L_U$  such that  $w_{L_U}(P) = n - 1$ . Then for each  $\mathbf{v} \in V \setminus W$  the  $\mathbb{F}_q$ -linear blocking set defined by  $\langle U, \mathbf{v} \rangle_{\mathbb{F}_q}$  has more than one Rédei line, each of them incident with  $P$ , and hence it is equivalent to the Rédei type blocking set obtained from  $\text{Tr}_{q^n/q}(x)$ .

Now let  $\mathcal{B}_1 = L_{V_1}$  and  $\mathcal{B}_2 = L_{V_2}$  be two  $\mathbb{F}_q$ -linear blocking sets of Rédei type with  $\text{PG}(W, \mathbb{F}_{q^n})$  the unique Rédei line. Denote by  $U_1$  and  $U_2$  the  $\mathbb{F}_q$ -subspaces  $W \cap V_1$  and  $W \cap V_2$ , respectively, and suppose  $L_{U_1} = L_{U_2}$  with  $\mathbb{F}_q$  the maximum field of linearity. Then  $\mathcal{B}_1$  and  $\mathcal{B}_2$  have  $(q + 1)$ -secants and we have  $V_1 = U_1 \oplus \langle \mathbf{u}_1 \rangle_{\mathbb{F}_q}$  and  $V_2 = U_2 \oplus \langle \mathbf{u}_2 \rangle_{\mathbb{F}_q}$  for some  $\mathbf{u}_1, \mathbf{u}_2 \in V \setminus W$ .

If  $\mathcal{B}_1^{\varphi^f} = \mathcal{B}_2$ , then [5, Proposition 2.3] implies  $V_1^f = \lambda V_2$  for some  $\lambda \in \mathbb{F}_{q^n}^*$ . Such  $f \in \Gamma\text{L}(3, q^n)$  has to fix  $W$  and it is easy to see that  $U_1^f = \lambda U_2$ , i.e.  $U_1$  and  $U_2$  are  $\Gamma\text{L}(2, q^n)$ -equivalent.

Conversely, if there exists  $f \in \Gamma\text{L}(W, \mathbb{F}_{q^n})$  such that  $U_1^f = U_2$ , then  $\mathcal{B}_1^{\varphi^g} = \mathcal{B}_2$ , where  $g \in \Gamma\text{L}(V, \mathbb{F}_{q^n})$  is the extension of  $f$  mapping  $\mathbf{u}_1$  to  $\mathbf{u}_2$ .  $\square$

## 5.4 Class of linear sets and MRD-codes

In [25, Section 4] Sheekey showed that maximum scattered linear sets of  $\text{PG}(1, q^n)$  correspond to  $\mathbb{F}_q$ -linear maximum rank distance codes (MRD-codes) of dimension  $2n$  and minimum distance  $n - 1$ , that is, a set  $\mathcal{M}$  of  $q^{2n}$   $n \times n$  matrices over  $\mathbb{F}_q$  forming an  $\mathbb{F}_q$ -subspace of  $\mathbb{F}_q^{n \times n}$  of dimension  $2n$  such that the non-zero matrices of  $\mathcal{M}$  have rank at least  $n - 1$ . For definitions and properties on MRD-codes we refer the reader to [9] by Delsarte and [12] by Gabidulin. For  $n \times n$  matrices there are two different definitions of

equivalence for MRD-codes in the literature. The arguments of [25, Section 4] yield the following interpretation of the  $\Gamma\text{L}$ -class:

- $\mathcal{M}$  and  $\mathcal{M}'$  are equivalent if there are invertible matrices  $A, B \in \mathbb{F}_q^{n \times n}$  and a field automorphism  $\sigma$  of  $\mathbb{F}_q$  such that  $A\mathcal{M}^\sigma B = \mathcal{M}'$ , see [25]. In this case the  $\Gamma\text{L}$ -class of  $L_U$  is the number of inequivalent MRD-codes obtained from the linear set  $L_U$ .
- $\mathcal{M}$  and  $\mathcal{M}'$  are equivalent if there are invertible matrices  $A, B \in \mathbb{F}_q^{n \times n}$  and a field automorphism  $\sigma$  of  $\mathbb{F}_q$  such that  $A\mathcal{M}^\sigma B = \mathcal{M}'$ , or  $A\mathcal{M}^{T^\sigma} B = \mathcal{M}'$ , see [8]. In this case the number of inequivalent MRD-codes obtained from the linear set  $L_U$  is between  $\lceil s/2 \rceil$  and  $s$ , where  $s$  is the  $\Gamma\text{L}$ -class of  $L_U$ .

We summarize here the known non-equivalent families of MRD-codes arising from maximum scattered linear sets.

1.  $L_{U_1} := \{\langle (x, x^q) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$  (found by Blokhuis and Lavrauw [4]) gives Gabidulin codes,
2.  $L_{U_2} := \{\langle (x, x^{q^s}) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$ ,  $\gcd(s, n) = 1$  ([4]) gives generalized Gabidulin codes,
3.  $L_{U_3} := \{\langle (x, \delta x^q + x^{q^{n-1}}) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$  (found by Lunardon and Polverino [21]) gives MRD-codes found by Sheekey,
4.  $L_{U_4} := \{\langle (x, \delta x^{q^s} + x^{q^{n-s}}) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$ ,  $N(\delta) \neq 1$ ,  $\gcd(s, n) = 1$  gives MRD-codes found by Lunardon, Trombetti and Zhou in [23].

**Remark 5.6.** *The linear sets  $L_{U_1}$  and  $L_{U_2}$  coincide, but when  $s \notin \{1, n-1\}$ , then there is no  $f \in \Gamma\text{L}(2, q^n)$  such that  $U_1^f = U_2$ . These linear sets are of pseudoregulus type, [19] (see also Example 5.1), and in [6] it was proved that the  $\Gamma\text{L}$ -class of these linear sets is  $\varphi(n)/2$ , hence they are examples of non-simple linear sets for  $n = 5$  and  $n > 6$ .*

It can be proved that the family  $L_{U_4}$  contains linear sets non-equivalent to those from the other families. We will report on this elsewhere.

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